Measuring tall tree heights from the ground

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Abstract

A range of trigonometric methods for measuring tall tree heights is described and reviewed. The Sine method using a range-finding laser is recommended as the simplest and most accurate technique, providing the top of the tree does not comprise fine dead branches, and there are clear lines of sight to the candidate high points. The Tan method, which triangulates the high point of the tree, is relatively complex but inherently accurate. The conventional method used in forestry, the Simple Tan, is not recommended for accurate estimation of tall tree heights unless it is used with surveyed plumb lines.

Introduction

Trees in excess of 80 m tall are amongst the world's largest organisms, and imbue in us feelings of awe and wonder. As national treasures, they are increasingly being catalogued (Carder 1995; Hickey *et al.* 2000) and their dimensions measured and recorded for posterity. These measurements should be as accurate as possible, using methodology that allows for precise re-measurement and estimation of error.

Besides being measured directly by tree climbers, heights of tall trees are often estimated trigonometrically from the ground using a method described in standard mensuration texts such as Philip (1994), Husch *et al.* (1982) and Carron (1968). With a clear line of sight, the measurer stands far enough from the tree so that the angle of elevation to the top does not exceed 45°, and the assistant positions a staff by the tree. If the tree leans, or the high point of the crown is eccentric to the stem, then the tree is measured at right angles to the plane of eccentricity. Measurements are made of elevation to the top, and angle and distance to the staff, and height is calculated trigonometrically from these measurements. This conventional method is referred to here as the Simple Tan method.

For tall trees, the Simple Tan method can be awkward and error-prone. Simply getting a clear line of sight to the base of a tree from 80 m or more may be difficult in heavily wooded or undulating terrain, but the difficulties are exacerbated if the tree leans or has a crown eccentricity that dictates the direction of measurement. The method does not easily allow multiple high points of a deliquescent (spreading) crown to be measured and compared because each high point requires a new measurement position at right angles to the eccentricity. The main drawback, however, is that the method is inherently imprecise unless the eccentricity is located by triangulation or plumb line.

This paper describes two trigonometric methods for estimating tree heights: the Sine method, recently made possible with the advent of range-finding lasers, and the Tan method, of which the Simple Tan method is a special case. Both the Sine and the Simple Tan methods have been used to measure tall trees in Tasmania, but as far as the author is aware, the Sine and Tan methods have not been fully described in the literature. Although tall trees present special challenges for accurate measurement, the methods described here apply to trees of all sizes.

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Definition of total height

Consider these definitions of total height for a tree:

1. Vertical distance from ground to stem tip.
2. Length of stem from stem base to stem tip.
3. Vertical distance from ground to the (possibly eccentric) high point.

The difference in these definitions may appear subtle, and for trees with vertical stems and pyramidal crowns, total height is the same using all definitions. However, for trees that lean, or have mature or deliquescent crowns, total height may vary considerably between definitions. So too do the practicalities and the achievable precision of measurement. Definitions 1 and 2 (Husch *et al.* 1982; Carron 1968) require measurement to a tip which is on the axis of the stem. This becomes a problem for mature and open crowns, because the tip must sometimes be regarded as an imaginary intersection of the general crown and the upward projection of the bole (Carron 1968), and measurement precision suffers accordingly. Definition 3 (Philip 1994) is more amenable to the measurement of mature and deliquescent crowns, and with some tightening of the definition (as follows) is the one used here.

As illustrated by Figure 1, tree height (h) is defined here as the vertical distance between two horizontal planes. The top plane ($\boldsymbol{P}_{\mathbf{T}}$) is positioned so that it is just touched by one point of the tree, and no other; it might be said that $\boldsymbol{P}_{\mathbf{T}}$ sits on the highest point, but 'highest' is a term I'm trying to define. The bottom plane ($\boldsymbol{P}_{\mathbf{B}}$) is positioned mid height between the upper and lower points ($\mathbf{S_U}$ and $\mathbf{S_L}$) of mineral soil around the trunk (avoiding buttresses and exposed roots) (Figure 2). $\mathbf{S_U}$ and $\mathbf{S_L}$ can be offset a short distance along the slope gradient to avoid litter and humus mounded at the base of the tree; however, it is important that the offset is identical for both.

This definition of height allows for the fact that the highest point of the tree may be a branch tip eccentric to the stem—a common occurrence in old deliquescent crowns. Note that the definition is not about length *per se*, and therefore leaning trees are disadvantaged.

The positioning of the top and bottom horizontal planes, $\boldsymbol{P}_{\mathbf{T}}$ and $\boldsymbol{P}_{\mathbf{B}}$, is debatable. The top of the tree could be regarded as the highest *live* point, thereby ignoring dead branches. And the bottom of the tree is often taken to be the highest point of mineral soil around the base, particularly in commercial forestry. However, it is recommended here that for the purposes of tall tree measurement, $\boldsymbol{P}_{\mathbf{T}}$ be positioned at the highest point of the tree, dead or alive, and $\boldsymbol{P}_{\mathbf{B}}$ be positioned *mid height* between $\mathbf{S_U}$ and $\mathbf{S_L}$, a practice commonly used for tall trees in the USA (Brett Mifsud, pers. comm. 2001).

Several other concepts are worth defining for this discussion:

1. The tree staff (with height h_T) is a graduated height pole situated at the base of the tree and sighted from measurement stations such as **V** (Figure 1). A second height pole may also be required; this is called the auxiliary pole.
2. The plumb line (**TG** in Figure 1) is the vertical line dropped from the high point of the tree to the ground. It plays a critical role in the Tan method.
3. High point eccentricity is defined as the horizontal distance between the base of the stem axis and the plumb line.
4. The first plane of eccentricity is the geometric plane defined by the plumb line and the base of the stem axis.
5. The second plane of eccentricity is the geometric plane defined by the plumb line and the tree staff. When the tree staff is in the first plane of eccentricity, then the two planes coincide. The term 'plane of eccentricity' used in this paper refers strictly to the second plane of eccentricity.

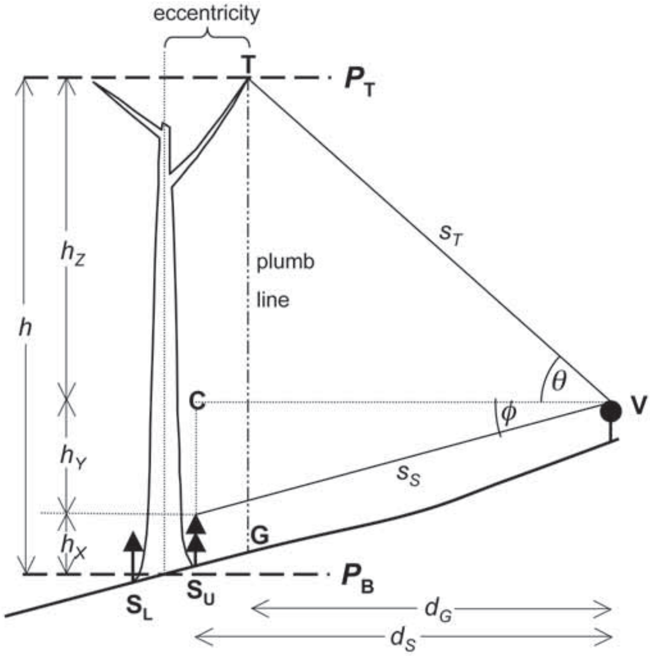

Figure 1. Notation for measurement of tree height.

Overview

Using notation shown in Figure 1, tree height (h) is calculated as:

[1] $h = h_X + h_Y + h_Z$

The Sine and Tan methods differ only in the way in which $\boldsymbol{h_Z}$ is estimated.

The Sine method uses:

[2] $h_Z = s_T \sin(\theta)$

where s_T, the slope distance from the measurement station (**V**) to the high point (**T**), is measured directly by a laser range-finder.

The Tan method uses:

[3] $h_Z = d_G \tan(\theta)$

where d_G, the horizontal distance from the measurement station (**V**) to the plumb line (**TG**), is estimated indirectly using surveying techniques.

Calculation of h_X—the bottom portion

$\boldsymbol{h_X}$ is the vertical distance from the top of the tree staff to the lower plane ($\boldsymbol{P_B}$) (Figure 1), and can be measured in several ways.

Option A

As illustrated by Figures 1 and 2, and preferred for uneven sloping ground:

1. Position the tree staff at $\mathbf{S_U}$ and the auxiliary pole at $\mathbf{S_L}$.
2. Position a theodolite (to the side of the tree) so that, when level, heights h_L and h_U can be read from the tree staff and the auxiliary pole respectively.

[4a] $h_X = h_T + \frac{h_L - h_U}{2}$

where h_T is the length of the tree staff.

Figures 1 and 2 and Equation 4a assume that the height pole at $\mathbf{S_U}$ is visible from the direction of the measurement station **V**. Where this is not the case, then the tree staff should be positioned at $\mathbf{S_L}$ and the auxiliary pole at $\mathbf{S_U}$, and heights h_L and h_U measured from the tree staff and auxiliary pole respectively. $\boldsymbol{h_X}$ is then calculated as:

[4b] $$h_X = h_T + \frac{h_U - h_L}{2}$$

Option B

The tree staff is positioned anywhere on the lower plane ($\boldsymbol{P_B}$). This is easily done when the ground is level. On uniformly sloping ground, the mid-height position is halfway between the contours containing $\mathbf{S_L}$ and $\mathbf{S_U}$. On uneven ground, however, this may not be true. $\boldsymbol{h_X}$ is simply the height of the tree staff:

[4c] $$h_X = h_T$$

Option C

This is an extension of Options A and B which allows the tree staff to be positioned off the lower plane ($\boldsymbol{P_B}$) anywhere around the tree. This might be needed by the Tan method which, for practical purposes, requires the tree staff to be on the same side of the tree as the plumb line.

1. Use Option A or B to calculate the vertical distance from the lower plane ($\boldsymbol{P_B}$) to the top of the tree staff. Previously called h_X, this distance is now the reference height h_{REF}.
2. Move the auxiliary pole to the position of the tree staff, and move the tree staff to the preferred position.
3. With a level theodolite, measure heights h_L and h_U on the auxiliary pole and tree staff respectively.

$\boldsymbol{h_X}$ is then calculated as:

[4d] $$h_X = h_{REF} + h_L - h_U$$

If the ground is level, then $h_X = h_{REF} = h_T$

Calculation of $\boldsymbol{h_Y}$—the middle portion

$\boldsymbol{h_Y}$ is the vertical distance from the measurement station **V** to the top of the tree staff, and is calculated as:

[5] $$h_Y = -s_S \sin(\phi)$$

where ϕ is the angle of depression from **V** to the tree staff, and s_S is the slope distance from **V** to the tree staff.

This paper uses the convention of negative depression and positive elevation. ϕ will normally be a depression (as shown in Figure 1) and therefore negative.

The positioning of measurement station **V** has important practical and theoretical ramifications. The angle of elevation (θ) from **V** to the top of the tree should be about 45° in order to minimise the impact of measurement error (see Appendix 1). Over this distance, which is about the height of the tree, a clear line of sight to the tree staff may not be possible and an intermediate measurement station may need to be used in order to use Equation 5 (see Appendix 2). For the Tan method, intermediate stations will need to be surveyed from **V** and the tree staff. However, the Sine method can avoid surveying intermediate stations by using a variant called the Compound Sine method.

Calculation of $\boldsymbol{h_Z}$—the top portion

The Sine method

The Sine method may have been used first in 1992 by Michael Taylor, an engineer and tall-tree enthusiast in the USA (Brett Mifsud, pers. comm. 2001). It estimates $\boldsymbol{h_Z}$ using Equation 2, where the slope distance (s_T) from **V** to the high point is measured directly by range-finding laser. The method is:

1. From **V**, measure the elevation (θ) and distance (s_T) to the high point (**T**) with a laser theodolite (or equivalent).
2. Calculate $\boldsymbol{h_Z}$.

Compared with the Tan method described below, it is simple, quick and much less prone to survey error. It automatically takes account of leaning trees and eccentric high points, thereby avoiding the cumbersome

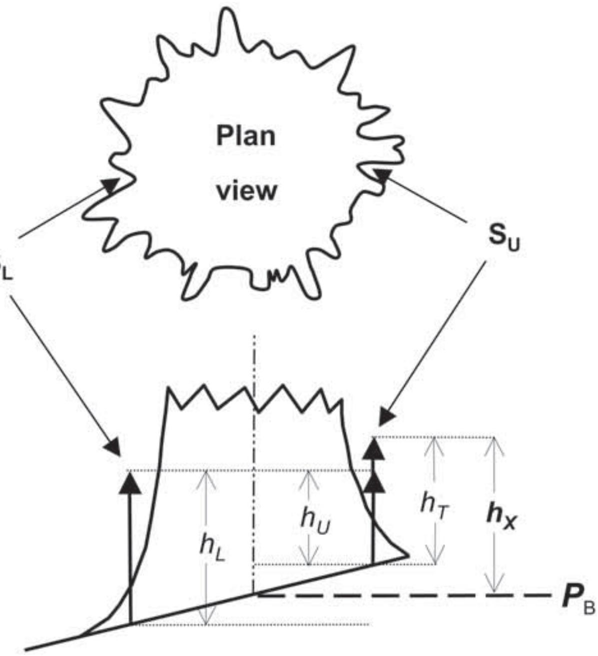

Figure 2. Calculation of $\boldsymbol{h_X}$.

task of surveying the plumb line or, in the case of the Simple Tan method, ensuring measurement at right angles to the plane of eccentricity.

The Sine method has two main drawbacks, both of which can lead to underestimation of height. Firstly, any obstruction by foreground leaves and branches is likely to result in the laser underestimating the distance to the intended target; there must be a clear line of sight to the target. The second drawback is laser 'insensitivity' to sparse distant twigs and small branches. Lasers vary in this regard depending upon their power, acquisition mode (pulse or phase) and processing algorithm. For each laser, there is a critical branch diameter (at a given distance) below which leafless branches will not be detected. If the top of the tree comprises dead branches, as is often the case for tall, fire-damaged or senescent eucalypts, then a laser may not detect the true high point if the branches are sufficiently small in diameter.

Some error can also be attributed to the laser range-finder's resolution of measurement. Distances displayed to the nearest one metre, for instance, will contribute up to ± 35 cm of error to the estimate of total height for each distance measured.

A well-behaved range-finder will not overestimate distance to a high point beyond its processing resolution (unless it detects a flock of birds) and, because of this, range-finders are sometimes used to get approximate tree heights by firing them at the top from near the base and 'fishing' for the biggest distance.

A useful variant of the Sine method, called the Compound Sine method (Figure 3), eliminates the need to see or survey the tree staff (**S_U**) from the measurement station (**V**). It does this by subdividing $\boldsymbol{h_Z}$ such that $h_Z = h_{Z1} + h_{Z2}$, and then estimating h_{Z1} and h_{Z2} from different measurement stations. Because the measurement stations need not be surveyed, the amount of scrub cutting and site disturbance is minimal.

As illustrated by Figure 3, the Compound Sine method is:

1. From **V**, locate an obvious point (**K**) on the tree such as a branch junction or burl.
2. From **V**, measure the elevation (θ) and slope distance (s_T) to the high point (**T**), and the elevation (θ_{K1}) and slope distance (s_{K1}) to **K**.
3. Establish a second measurement station (**W**) nearer the tree and within sight of both **K** and **S_U**.
4. From **W**, measure the elevation (θ_{K2}) and slope distance (s_{K2}) to **K**, and the angle of depression (ϕ) and slope distance (s_S) to **S_U**.
5. Calculate $\boldsymbol{h_Z}$.

$$h_Z = h_{Z1} + h_{Z2}$$
$$= [s_T \sin(\theta) - s_{K1} \sin(\theta_{K1})] + [s_{K2} \sin(\theta_{K2})]$$

THE TAN METHOD

The Tan method estimates $\boldsymbol{h_Z}$ using Equation 3, where d_G is the horizontal distance from **V** to the plumb line (**TG**) (Figure 1). Surveying a plumb line can be an exacting and time-consuming process, and one will need to be established for each candidate high point. Three alternatives are described here; they are labelled D, E and F so that they are not confused with alternatives for measuring $\boldsymbol{h_X}$.

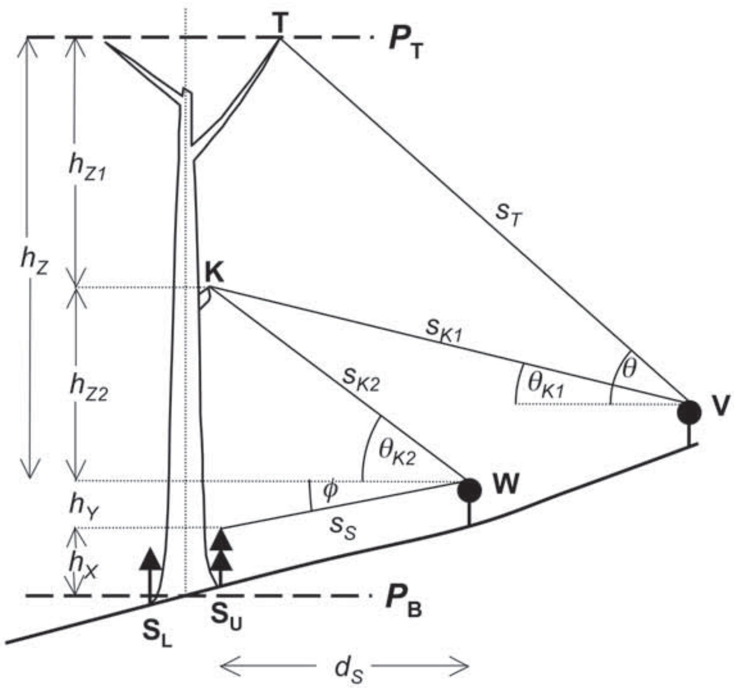

Figure 3. Compound Sine method.

Option D—'Opposite' base line method (Figure 4a)

1. Establish a second measurement station **W** so that the angle $\angle VTW$ is approximately 90°, at a distance at least half tree height from the tree.
2. From **V** survey **W**, recording bearing (β_{VW}), elevation (ϕ_{VW}) and slope distance (s_{VW}). If not visible, then a chain of sub-stations may need to be surveyed (see Appendix 2).
3. From **V**, record the bearing (β_{VT}) of the high point (**T**).
4. From **W**, record the bearing (β_{WT}) of the high point (**T**).
5. Calculate the horizontal length of the base line (d_B) and the internal angles (shown in Example 1, based on Figure 4a).
6. Calculate d_G (Example 2).

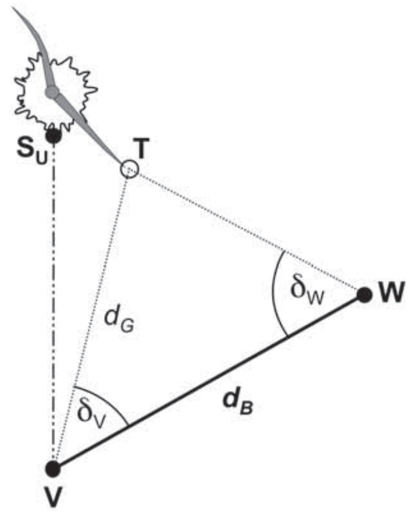

Figure 4a. Triangulation with 'opposite' base line.

Example 1

$d_B = s_{VW}\cos(\phi_{VW})$	$d_B = 89.25\cos(4) = 89.03$
$\beta_{WV} = \mathrm{mod}(180 + \beta_{VW}, 360)$	$\beta_{WV} = \mathrm{mod}(180 + 61, 360) = 241$
$\delta_W = f(\lvert\beta_{WV} - \beta_{WT}\rvert)$	$\delta_W = f(\lvert 241 - 298\rvert) = 57$
$\delta_V = f(\lvert\beta_{VW} - \beta_{VT}\rvert)$	$\delta_V = f(\lvert 61 - 14\rvert) = 47$

where $f(\beta) = \begin{cases} \beta & if\ \beta < 180 \\ 360 - \beta & if\ \beta \geq 180 \end{cases}$

Example 2

$d_G = \dfrac{d_B \sin(\delta_W)}{\sin(\delta_W + \delta_V)}$	$d_G = \dfrac{89.03\sin(57)}{\sin(57 + 47)} = 76.95$

Option E—'Adjacent' base line method (Figure 4b)

This may be simpler to lay out than Option D, but more difficult to calculate d_G.

1. Establish a second measurement station **W** so that the angle $\angle VTW$ is approximately 90°, and at a distance at least half tree height from the tree.
2. From the tree staff $\mathbf{S_U}$ survey **W**, recording elevation (ϕ_{SW}) and slope distance (s_{SW}).
3. From **W**, record the bearing (β_{WT}) of the high point (**T**), and the bearing (β_{WS}) to the tree staff ($\mathbf{S_U}$).
4. From **V**, record the bearing (β_{VT}) of the high point (**T**), and the bearing (β_{VS}) to the tree staff ($\mathbf{S_U}$).
5. Calculate the horizontal length of the base lines and the internal angles (shown in Example 3), based on Figures 1 and 4b.
6. Calculate d_B (Example 4).
7. Calculate internal angles σ_V and σ_W (Example 5).
8. Calculate d_G (Example 6).

Example 3

$d_S = s_S \cos(\phi)$	$d_S = 85.30 \cos(11) = 83.73$
$d_V = s_{SW} \cos(\phi_{SW})$	$d_V = 89.24 \cos(15) = 86.20$
$\lambda_S = f(\lvert\beta_{WS} - \beta_{VS}\rvert)$	$\lambda_S = f(\lvert 270 - 0\rvert) = 90$
$\lambda_W = f(\lvert\beta_{WS} - \beta_{WT}\rvert)$	$\lambda_W = f(\lvert 270 - 262.35\rvert) = 7.65$
$\lambda_V = f(\lvert\beta_{VS} - \beta_{VT}\rvert)$	$\lambda_V = f(\lvert 0 - 14\rvert) = 14$

where $f(.)$ is as defined above under option D.

Example 4

$d_B = \sqrt{d_S^2 + d_V^2 - 2 d_S d_V \cos(\lambda_S)}$	$d_B = \sqrt{83.73^2 + 86.20^2 - 2 * 83.73 * 86.20 \cos(90)} = 120.17$

Example 5

$\sigma_V = \arcsin\left(\frac{d_V \sin(\lambda_S)}{d_B}\right) - \lambda_V$	$\sigma_V = \arcsin\left(\frac{86.20 \sin(90)}{120.17}\right) - 14 = 31.83$
$\sigma_W = \arcsin\left(\frac{d_S \sin(\lambda_S)}{d_B}\right) - \lambda_W$	$\sigma_W = \arcsin\left(\frac{83.73 \sin(90)}{120.17}\right) - 7.65 = 36.52$

Example 6

$d_G = \frac{d_B \sin(\sigma_W)}{\sin(\sigma_W + \sigma_V)}$	$d_G = \frac{120.17 \sin(36.52)}{\sin(36.52 + 31.83)} = 76.94$

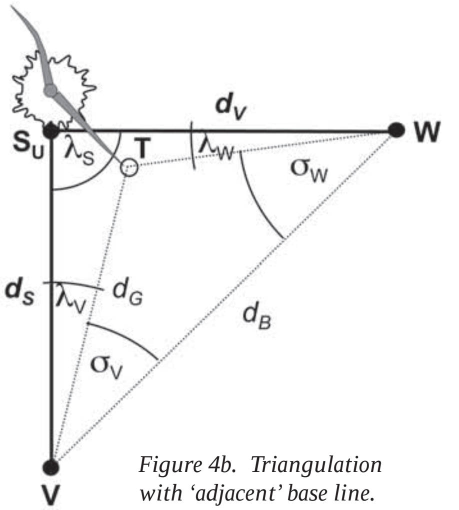

Figure 4b. Triangulation with 'adjacent' base line.

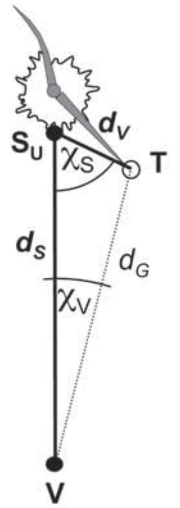

Figure 4c. Triangulation with the 'Plumb-Bob' method.

Option F—The 'Plumb-Bob' method (Figure 4c)

This is described in the Australian and New Zealand *Code of Mensuration Practice* (Research Working Group No. 2, 1999) and, depending on the circumstances, may be the easiest of the triangulation options. If a pole positioned at the base of the plumb line (**TG**) is visible from both the tree staff and **V** (or a sub-station), then the method is easy. Otherwise, it is awkward and offsets will be required. Plumb lines internal to the trunk also require offsets so that a height pole can be positioned external to the tree. Options D and E do not suffer from these problems.

The simple case where the plumb line is external to the trunk and is visible from both the tree staff and **V**, is described as follows:

1. Using a three-person crew, position two plumb-bobs at right angles to the tree, each at about half tree height distance, and simultaneously position a pole at the base of the plumb line.
2. From the plumb line (**T**), survey the tree staff (**S_U**), recording elevation (ϕ_{TS}), slope distance (s_{TS}) and bearing (β_{TS}).
3. From **V**, record the bearing (β_{VT}) of the high point (**T**), and the bearing (β_{VS}) to the tree staff.
4. Calculate internal angles and horizontal distances (shown in Example 7, based on Figure 4c).
5. Calculate d_G (Example 8).

Example 7

$\chi_S = f(\lvert\beta_{VS} - \beta_{TS}\rvert)$	$\chi_S = f(\lvert 0 - 296\rvert) = 64$
$\chi_V = f(\lvert\beta_{VS} - \beta_{VT}\rvert)$	$\chi_V = f(\lvert 0 - 14\rvert) = 14$
$d_V = s_{TS}\cos(\phi_{TS})$	$d_V = 21.03\cos(10) = 20.71$

where *f*(.) is as defined above under Option D.

Example 8

$d_G = \dfrac{d_V \sin(\chi_S)}{\sin(\chi_V)}$	$d_G = \dfrac{20.71\sin(64)}{\sin(14)} = 76.94$

The Simple Tan method

The Simple Tan method is the conventional method for heighting trees in commercial forestry, and simplifies the general Tan method by making two assumptions:

1. That the plumb line can be established by eye; and
2. That the measurement station (**V**) is equidistant from the (imaginary) plumb line and the tree staff.

When these assumptions hold true, then the Simple Tan method is accurate. But for tall trees in particular, the plumb line is difficult to estimate by eye without using one of the survey methods described above.

This method also uses Equation 3 to calculate $\boldsymbol{h_Z}$. Because of assumption 2, however, the horizontal distance from **V** to the tree staff ($d_S = s_S \cos(\phi)$) is used instead of the horizontal distance to the plumb line (d_G). Non-compliance with the assumption that $d_S = d_G$ is the main source of error with the Simple Tan. If the elevation (θ) to the high point (**T**) is 45°, then the error in height (ξ) is:

$$\xi = d_S - d_G$$

Errors of this type can be large, and are likely to produce over-estimates of tree height because of the tendency for false or apparent high points to occur in the front half of the crown.

It should be noted that for perfectly straight trees with no high point eccentricity, use of a tree staff creates a second plane of eccentricity. Of course, **V** should be positioned perpendicular to this plane at a point mid-way between the tree staff and the plumb line, which simply means that the staff should be positioned to the side of the tree when looked at from **V**. This is standard practice for Simple Tan measurement.

Conclusion

Provided there is a clear line of sight and the top of the tree is defined by foliage or substantial dead branches, the Sine method is quicker, simpler and less prone to survey error than the Tan method. It is ideal for measuring candidate high points from a single location in order to find the highest. Where the tree staff is not visible from the measurement station, establishment of a surveyed intermediate station can be avoided by using the Compound Sine method. Error will occur when the top of the tree comprises sparse dead branches which are too fine for the laser to register, although this should become less of a problem as laser instruments improve. By and large, errors associated with the Sine method will underestimate tree height.

The Tan method can be used in every situation, but triangulation of the plumb line is reasonably complex and therefore prone to error, and where the tree staff is not visible from the measurement station, establishment of a surveyed intermediate station is obligatory. However, once a base line has been established (Options A and B), measurement of multiple high points is straightforward as long as all high points are visible from both ends of the base line. If the survey work is highly accurate, then the Tan method can be more accurate than the Sine method because it uses optical processes which are relatively immune to foreground interference and branch size.

The conventional Simple Tan method should only be used to measure tall trees if a plumb line is surveyed for each candidate high point.

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Appendix 1. Preferred angle of elevation

Provided that trigonometric assumptions are correct, error in tree height (ξ) is due to errors in the measurement of distance and angles. If we assume that erroneous angles are independent of elevation (θ) and contribute a relatively large proportion of ξ, then ξ can be defined as:

[1] $$\xi = h\left(\frac{\tan(\theta+\Delta)}{\tan(\theta)} - 1\right)$$

where ξ is error in tree height, h is true tree height, θ is the true angle of elevation, and Δ is the error in θ.

ξ is minimised by minimising:

$$\Omega = \frac{\tan(\theta+\Delta)}{\tan(\theta)}$$

which is achieved by finding θ^* such that $\frac{d\Omega}{d\theta} = 0$. It follows that:

$$\frac{d\Omega}{d\theta} = \frac{\sec^2(\theta^*+\Delta)}{\tan(\theta^*)} - \frac{\tan(\theta+\Delta)\sec^2(\theta^*)}{\tan^2(\theta^*)} = 0$$

$$\Rightarrow \tan(\theta^*)\sec^2(\theta^*+\Delta) = \tan(\theta^*+\Delta)\sec^2(\theta^*)$$

$$\Rightarrow \sin(\theta^*)\cos(\theta^*) = \sin(\theta^*+\Delta)\cos(\theta^*+\Delta)$$

$$\Rightarrow \sin(2\theta^*) = \sin(2\theta^*+2\Delta) = \sin(2\theta^*)\cos(2\Delta) + \cos(2\theta^*)\sin(2\Delta)$$

$$\Rightarrow 1 = \cos(2\Delta) + \cot(2\theta^*)\sin(2\Delta)$$

$$\Rightarrow \theta^* = 0.5\arctan\left(\frac{\sin(2\Delta)}{1-\cos(2\Delta)}\right)$$

$$= 0.5\arctan(\cot(\Delta))$$

$$= 0.5(90-\Delta)$$

$$= 45 - \frac{\Delta}{2}$$

ξ is fairly insensitive to changes in θ in the vicinity of θ^*. For all practical Δ, ξ is increased by less than 1% for θ in the range of $\theta^* \pm 4°$. Therefore, 45° can be recommended as the preferred angle of elevation. Of course, it is also the preferred angle of depression (to the bottom of the tree) but there is much less flexibility with these angles because they are generally not affected by distance from the tree.

Appendix 2. Traverse of Stations

It may not be possible to see the base of a tall tree from the primary measurement station **V**, so bearing (β), elevation (ϕ) and slope distance (s) to the bottom of the tree cannot be measured directly. In that case, a chain of stations must be established and surveyed. Surveyors call this a 'traverse'. Bearing, elevation and slope distance from each station to the next must be measured, culminating with the destination height pole (or tree staff). Care needs to be taken in correcting for differences in height of the height pole and the measurement instrument when subsequently positioned at the same point.

Notation (as per Figure A2.1)

β_i is the bearing from the ith to the (i+1)th station.
ϕ_i is the elevation from the ith to the (i+1)th station. Make depression negative.
s_i is the slope distance from the ith to the (i+1)th station.
t_i is the height of the ith measurement station.
p_i is the height of the ith height pole.

$$\varepsilon_{i+1} = \begin{cases} p_{i+1} - t_i & \\ 0 & \text{when } p_{i+1} \text{ is the tree staff.} \end{cases}$$

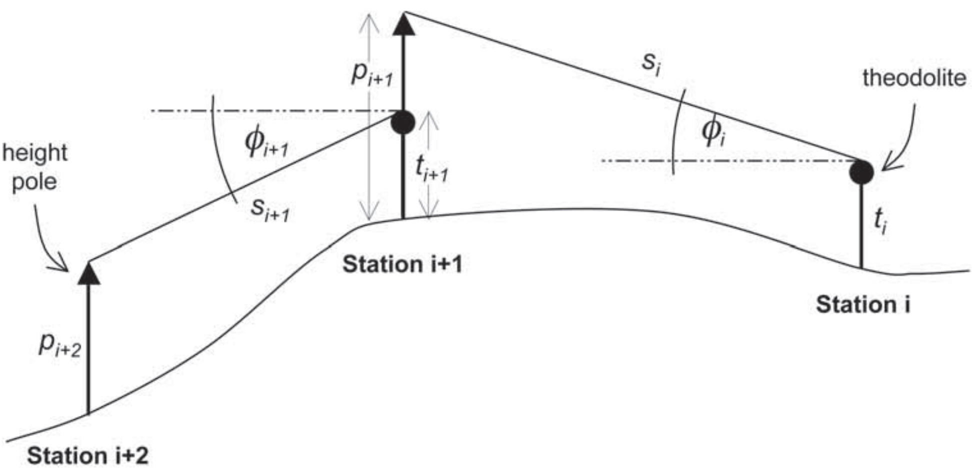

Figure A2.1. Notation for station traverse.

Method

1. Set up a theodolite at a predetermined position and measure its height (t_i). This is station '*i*'. It will either be the start of the traverse (**V**), or at the location of the height pole from the previous measurement.
2. Position a height pole some distance from the theodolite in the direction of the traverse, and mark its position on the ground. This will become station '*i+1*'.
3. Measure bearing (β_i), elevation (ϕ_i) and slope distance (s_i) to the height pole, and the height of the pole (p_{i+1}).
4. Repeat steps 1 to 3 until the height pole is at the destination of the traverse.

Calculations

The overall bearing (β), elevation (ϕ) and slope distance (s) from the first measurement station to the last height pole are calculated as follows:

$$\beta = \arctan\left(\frac{x}{z}\right)$$

$$\phi = \arctan\left(\frac{y}{\sqrt{x^2 + z^2}}\right)$$

$$s = \sqrt{x^2 + y^2 + z^2}$$

where:

$$y = \sum_i s_i.\sin(\phi_i - \Delta_i)$$

$$x = \sum_i s_i.\cos(\phi_i - \Delta_i).\sin(\beta_i)$$

$$z = \sum_i s_i.\cos(\phi_i - \Delta_i).\cos(\beta_i)$$

$$\Delta_i = \arcsin\left(\frac{\varepsilon_{i+1}}{s_i}\right)$$

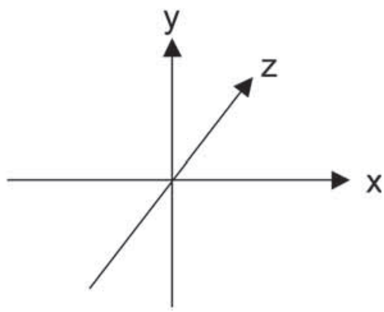

Figure A2.2. Orientation of 3D axes used in notation.

NOTE: $\frac{\varepsilon_{i+1}}{s_i}$ must be less than 0.05

Tasforests